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Discrete Conjugate Boundary Value Problems

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Abstract—This paper discusses higher-order discrete conjugate boundary value problems of singular and nonsingular type.

Keywords—Conjugate discrete boundary value problems, Singular problems, Nonsingular problems, Nonnegative solutions.

1. INTRODUCTION

This paper discusses the n^{th} $(n \geq 2)$ order discrete conjugate boundary value problem

$$\begin{aligned} (-1)^{n-p} \Delta^n y(i-p) &= f(i, y(i)), \qquad i \in J_p, \\ \Delta^i y(0) = 0, &\qquad 0 \leq i \leq p-1 \text{ (i.e., } y(0) = \cdots = y(p-1) = 0), \\ \Delta^i y(T+n-i) = 0, &\qquad 0 \leq i \leq n-p-1 \text{ (i.e., } y(T+p+1) = \cdots = y(T+n) = 0); \end{aligned} \tag{1.1}$$

here $T \in \{1, 2, \dots\}$, $J_p = \{p, p+1, \dots, T+p\}$, $1 \leq p \leq n-1$, and $y : I_n = \{0, 1, \dots, T+n\} \to \mathbf{R}$. We will let $C(I_n)$ denote the class of maps w continuous on I_n (discrete topology) with norm $\|w\| = \max_{k \in I_n} |w(k)|$. By a solution to (1.1), we mean a $w \in C(I_n)$ such that w satisfies the difference equation in (1.1) for $i \in J_p$ and w satisfies the conjugate boundary data. In this paper, we discuss separately the cases when f is nonsingular and when f is singular (i.e., $f(i,u)$ singular at $u = 0$). All the results are new and they extend and complement the theory in the literature [1–5]. Indeed this paper is the first time the singular higher-order discrete conjugate boundary value problem has been discussed (see [6] for results when $n = 2$).

For the remainder of this introduction, we gather together some results which will be used in Sections 2 and 3. In [1–3,5], it was shown that if y satisfies

$$\begin{aligned} \Delta^n y(k) &= \phi(k), \qquad k \in I_0 = \{0, 1, \dots, T\}, \\ \Delta^i y(0) &= 0, \qquad 0 \leq i \leq p-1, \\ \Delta^i y(T+n-i) &= 0, \qquad 0 \leq i \leq n-p-1, \end{aligned} \tag{1.2}$$

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then

$$y(k) = \sum_{j=0}^{T} G_3(k,j)\ \phi(j), \qquad \text{for } k \in I_n;$$

here

$$G_3(k,j) = \sum_{l=0}^{p-1} \left[\sum_{i=0}^{p-l-1} \binom{n-p+i-1}{i} \frac{k^{(l+i)}}{(T+n-l)^{(n-p+i)}} \right] \frac{(-j-1)^{(n-l-1)}}{l!\,(n-l-1)!} (T+n-k)^{(n-p)}$$

if $j \in \{0, 1, ..., k-1\}$, and

$$G_3(k,j) = -\sum_{l=0}^{n-p-1} \left[\sum_{i=0}^{n-p-l-1} \binom{p+i-1}{i} \frac{(T+p+l+i-k)^{(l+i)}}{(T+p+1+l+i)^{(p+i)}} \right] \frac{(-1)^l\,(T+p-j)^{(n-l+1)}\,k^{(p)}}{l!\,(n-l-1)!}$$

if $j \in \{k, k+1, \dots, T\}$. Next consider

$$\begin{aligned} \Delta^n y(k-p) &= \phi(k), && k \in J_p, \\ \Delta^i y(0) &= 0, && 0 \le i \le p-1, \\ \Delta^i y(T+n-i) &= 0, && 0 \le i \le n-p-1. \end{aligned} \tag{1.3}$$

Notice (1.3) is the same as

$$\begin{aligned} \Delta^n y(k) &= \phi(k+p), && k \in I_0, \\ \Delta^i y(0) &= 0, && 0 \le i \le p-1, \\ \Delta^i y(T+n-i) &= 0, && 0 \le i \le n-p-1, \end{aligned} \tag{1.4}$$

and so

$$y(k) = \sum_{j=0}^{T} G_3(k,j)\ \phi(j+p), \qquad \text{for } k \in I_n.$$

This is the same as

$$y(k) = \sum_{j=p}^{T+p} G_3(k,j-p)\ \phi(j), \qquad \text{for } k \in I_n. \tag{1.5}$$

We write

$$y(k) = \sum_{j=p}^{T+p} G(k,j)\ \phi(j), \qquad \text{for } k \in I_n, \tag{1.6}$$

where

$$G(k,j) = G_3(k,j-p). \tag{1.7}$$

Also in [2,3], the following result was established.

THEOREM 1.1. *Suppose $y : I_n \to \mathbf{R}$ is such that*

$$\begin{aligned} (-1)^{n-p}\,\Delta^n y(k) &\ge 0, && k \in I_0, \\ \Delta^i y(0) &= 0, && 0 \le i \le p-1, \\ \Delta^i y(T+n-i) &= 0, && 0 \le i \le n-p-1. \end{aligned} \tag{1.8}$$

Then

$$y(k) \ge \theta \max_{j \in I_n} |y(j)| = \theta\ \|y\|, \qquad \textit{for } k \in J_p, \tag{1.9}$$

where $0 < \theta < 1$ *is a constant given by*

$$\theta = \min\{b(p),\, b(p+1)\}; \tag{1.10}$$

here b *is given by*

$$b(x) = \frac{\min\{g(x,p),\, g(x,T+p)\}}{\min\{g(x,[\theta(x)]), g(x,[\theta(x)+1]), g(x,p),\, g(x,T+p)\}} \tag{1.11}$$

with

$$g(x,k) = k^{(x-1)}\,(T+n-k)^{(n-x)} \quad \textit{and} \quad \theta(x) = \frac{(x-1)T + (x-2)n + x}{n-1} \tag{1.12}$$

(note $[\,.\,]$ *denotes the greatest integer function).*

Now suppose $y : I_n \to \mathbf{R}$ satisfies

$$\begin{aligned} (-1)^{n-p}\,\Delta^n y(k-p) &\geq 0, && k \in J_p,\\ \Delta^i y(0) &= 0, && 0 \leq i \leq p-1,\\ \Delta^i y(T+n-i) &= 0, && 0 \leq i \leq n-p-1. \end{aligned} \tag{1.13}$$

Of course, $(-1)^{n-p}\,\Delta^n y(k-p) \geq 0$ for $k \in J_p$ is exactly the same as $(-1)^{n-p}\,\Delta^n y(k) \geq 0$ for $k \in I_0$ and so

$$y(k) \geq \theta\, \|y\| = \theta \max_{j \in I_n} |y(j)|, \qquad \text{for } k \in J_p. \tag{1.14}$$

Next, we present an existence principle for the discrete conjugate boundary value problem

$$\begin{aligned} (-1)^{n-p}\,\Delta^n y(k-p) &= f(k, y(k)), && k \in J_p,\\ y(0) &= a,\\ \Delta^i y(0) &= 0, && 1 \leq i \leq p-1,\\ y(T+n) &= a,\\ \Delta^i y(T+n-i) &= 0, && 1 \leq i \leq n-p-1. \end{aligned} \tag{1.15}$$

THEOREM 1.2. *Suppose* $f : J_p \times \mathbf{R} \to \mathbf{R}$ *is continuous (i.e., continuous as a map from the topological space* $J_p \times \mathbf{R}$ *into the topological space* $\mathbf{R}$ *(of course, the topology on* J_p *will be the discrete topology)). Assume there is a constant* $M > |a|$*, independent of* λ*, with*

$$\|y\| = \max_{j \in I_n} |y(j)| \neq M$$

for any solution $y \in C(I_n)$ *to*

$$\begin{aligned} (-1)^{n-p}\,\Delta^n y(k-p) &= \lambda\, f(k, y(k)), && k \in J_p,\\ y(0) &= a,\\ \Delta^i y(0) &= 0, && 1 \leq i \leq p-1,\\ y(T+n) &= a,\\ \Delta^i y(T+n-i) &= 0, && 1 \leq i \leq n-p-1, \end{aligned} \tag{1.16$_\lambda$}$$

for each $\lambda \in (0,1)$*. Then (1.15) has a solution.*

PROOF. Solving $(1.16)_\lambda$ is equivalent to finding a $y \in C(I_n)$ which satisfies

$$y(k) = a + \lambda \sum_{j=p}^{T+p} G(k,j)\, f(j, y(j)), \qquad \text{for } k \in I_n; \tag{1.17$_\lambda$}$$

here G is as in (1.7). Define the operator $S : C(I_n) \to C(I_n)$ by setting

$$S\,y(k) = a + \sum_{j=p}^{T+p} G(k,j)\; f(j, y(j)).$$

Now (1.17)

$$y = (1-\lambda)a + \lambda\, S\, y.$$

It is easy to see [3,7] that $S : C(I_n) \to C(I_n)$ is continuous and completely continuous. Let

$$U = \{u \in C\,(I_n) : \|u\| < M\} \quad \text{and} \quad E = C\,(I_n)\,.$$

The nonlinear alternative of Leray-Schauder [4] guarantees that S has a fixed point in $\overline{U}$, i.e., (1.15) has a solution. ∎

2. NONSINGULAR PROBLEMS

In this section, we are interested in establishing the existence of nonnegative solutions to discrete conjugate higher-order boundary value problems of nonsingular type. For convenience we discuss (1.1). However, we note that the ideas in *this* section could be used to discuss other higher-order discrete problems; for example, the (n,p), focal, and conjugate problems in [3].

THEOREM 2.1. *Suppose the following conditions are satisfied:*

$$f : J_p \times [0,\infty) \to [0,\infty) \text{ is continuous,} \tag{2.1}$$

there exists a continuous, nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ with $\psi > 0$ on $(0,\infty)$ and a function $q : J_p \to [0,\infty)$ with $f(k,u) \le q(k)\,\psi(u)$, for all $u \ge 0$ and $k \in J_p$ (2.2)

and

$$\sup_{c\in(0,\infty)} \left(\frac{c}{\psi(c)}\right) > Q; \qquad \text{here } Q = \max_{k\in I_n} \sum_{j=p}^{T+p} q(j)\,(-1)^{n-p}\, G(k,j). \tag{2.3}$$

Then (1.1) has a nonnegative solution.

PROOF. Consider the family of problems

$$\begin{aligned} (-1)^{n-p}\,\Delta^n y(k-p) &= \lambda\, f^\star(k, y(k)), && k \in J_p,\\ \Delta^i y(0) &= 0, && 0 \le i \le p-1,\\ \Delta^i y(T+n-i) &= 0, && 0 \le i \le n-p-1, \end{aligned} \tag*{$(2.4)_\lambda$}$$

for $0 < \lambda < 1$; here

$$f^\star(k,u) = \begin{cases} f(k,u), & u \ge 0,\\ f(k,0), & u \le 0. \end{cases}$$

Let y be any solution of $(2.4)_\lambda$ for $0 < \lambda < 1$. Then

$$y(k) = \lambda \sum_{j=p}^{T+p} (-1)^{n-p}\, G(k,j)\, f(j, y(j)) \ge 0, \qquad \text{for } k \in I_n. \tag{2.5}$$

For notational purposes let $y_0 = \sup_{k\in I_n} y(k)$. Let $M > 0$ satisfy

$$\frac{M}{\psi(M)} > Q. \tag{2.6}$$

From (2.5) we have for $k \in I_n$,

$$y(k) \le \sum_{j=p}^{T+p} (-1)^{n-p}\, G(k,j)\, q(j)\, \psi(y(j)) \le \psi(y_0) \sum_{j=p}^{T+p} (-1)^{n-p}\, G(k,j)\, q(j) \le Q\, \psi\,(y_0)$$

and so

$$\frac{y_0}{\psi\,(y_0)} \le Q. \tag{2.7}$$

Now (2.6) together with (2.7) implies $y_0 \ne M$. Thus, any solution y of $(2.4)_\lambda$ satisfies $\|y\| \ne M$, i.e., $y_0 \ne M$. Now Theorem 1.2 implies $(2.4)_1$ has a solution y and, of course, $y(k) \ge 0$ for $k \in I_n$. Thus, y is a solution of (1.1). ∎

3. SINGULAR PROBLEMS

Next we discuss (1.1) when our nonlinearity $f(i,y)$ may be singular at $y = 0$.

THEOREM 3.1. *Suppose the following conditions are satisfied:*

$$f : J_p \times (0,\infty) \to (0,\infty) \text{ is continuous,} \tag{3.1}$$

$$\begin{gathered} f(k,u) \le g(u) + h(u) \text{ on } J_p \times (0,\infty) \text{ with } g > 0 \text{ continuous and nonincreasing} \\ \text{on } (0,\infty),\ h \ge 0 \text{ continuous on } [0,\infty) \text{ and } h/g \text{ nondecreasing on } (0,\infty) \end{gathered} \tag{3.2}$$

$$\begin{gathered} \text{for each constant } H > 0 \text{ there exists a continuous function} \\ \psi_H : J_p \to (0,\infty) \text{ with } f(k,u) \ge \psi_H(k) \text{ on } J_p \times (0,H] \end{gathered} \tag{3.3}$$

$$\begin{gathered} \text{there exists a constant } K_\theta > 0 \text{ with } g(\theta\, u) \le K_\theta\, g(u) \\ \text{for all } u \ge 0;\text{ here } \theta \text{ is as in (1.10)} \end{gathered} \tag{3.4}$$

and

$$\sup_{c \in (0,\infty)} \left(\frac{c}{g(c) + h(c)} \right) > K_\theta\, Q; \tag{3.5}$$

here

$$Q = \max_{k \in J_p} \sum_{j=p}^{T+p} (-1)^{n-p}\, G(k,j) \text{ and } G \text{ is as in (1.7).}$$

Then (1.1) has a solution $y \in C(I_n)$ with $y(i) > 0$ for $i \in J_p$.

PROOF. Choose $M > 0$ with

$$\frac{M}{Q\, K_\theta\, [g(M) + h(M)]} > 1. \tag{3.6}$$

Next choose $\epsilon > 0$ and $\epsilon < M$ with

$$\frac{M}{Q\, K_\theta\, [g(M) + h(M)] + \epsilon} > 1. \tag{3.7}$$

Let $n_0 \in \{1,2,\dots\}$ be chosen so that $1/n_0 < \epsilon$ and let $N_0 = \{n_0, n_0+1, \dots\}$. We first show

$$\begin{aligned} (-1)^{n-p}\, \Delta^n\, y(k-p) &= f^{**}(k, y(k)), && k \in J_p, \\ y(0) &= \frac{1}{m}, && \\ \Delta^i\, y(0) &= 0, && 1 \le i \le p-1, \\ y(T+n) &= \frac{1}{m}, && \\ \Delta^i\, y(T+n-i) &= 0, && 1 \le i \le n-p-1, \end{aligned} \tag*{$(3.8)^m$}$$

has a solution for each $m \in N_0$; here

$$f^{**}(k,u) = \begin{cases} f(k,u), & u \geq \dfrac{1}{m}, \\ f\left(k, \dfrac{1}{m}\right), & u < \dfrac{1}{m}. \end{cases}$$

To show $(3.8)^m$ has a solution for each $m \in N_0$, we will apply Theorem 1.2. Consider the family of problems

$$\begin{aligned} (-1)^{n-p}\,\Delta^n y(k-p) &= \lambda\, f^{**}(k, y(k)), && k \in J_p, \\ y(0) &= \frac{1}{m}, \\ \Delta^i y(0) &= 0, && 1 \leq i \leq p-1, \\ y(T+n) &= \frac{1}{m}, \\ \Delta^i y(T+n-i) &= 0, && 1 \leq i \leq n-p-1, \end{aligned} \tag{3.9$_\lambda^m$}$$

for $0 < \lambda < 1$. Let $y \in C(I_n)$ be any solution of $(3.9)_\lambda^m$. Then

$$y(k) = \frac{1}{m} + \lambda \sum_{j=p}^{T+p} (-1)^{n-p}\, G(k,j)\, f^{**}(j, y(j)), \qquad \text{for } k \in I_n, \tag{3.10}$$

and so $y(k) \geq 1/m$ for $k \in I_n$.

REMARK 3.1. Any solution u of $(3.9)_1^m$ satisfies $u(k) \geq 1/m$ for $k \in I_n$ also.

We next *claim* that

$$\|y\| = \sup_{j \in I_n} y(j) \neq M, \qquad \text{(here } M \text{ is as in (3.6))} \tag{3.11}$$

for any solution y to $(3.9)_\lambda^m$. To see this, let y be any solution of $(3.9)_\lambda^m$ and let the absolute maximum of $y(k)$ be at say $i_0 \in J_p$. Then (3.10), (3.2), (1.14), and (3.4) (with $y(k) \geq 1/m$ for $k \in I_n$) imply

$$\begin{aligned} y(i_0) &\leq \frac{1}{m} + \left\{1 + \frac{h(y(i_0))}{g(y(i_0))}\right\} \sum_{j=p}^{T+p} (-1)^{n-p}\, G(i_0, j)\; g(y(j)) \\ &\leq \epsilon + \left\{1 + \frac{h(y(i_0))}{g(y(i_0))}\right\} \sum_{j=p}^{T+p} (-1)^{n-p}\, G(i_0, j)\; g(\theta\, y(i_0)) \\ &\leq \epsilon + [g(y(i_0)) + h(y(i_0))]\; K_\theta \sum_{j=p}^{T+p} (-1)^{n-p}\, G(i_0, j) \\ &\leq \epsilon + [g(y(i_0)) + h(y(i_0))]\; K_\theta\, Q. \end{aligned}$$

Consequently,

$$\frac{y(i_0)}{\epsilon + [g(y(i_0)) + h(y(i_0))]\; K_\theta\, Q} \leq 1. \tag{3.12}$$

Now (3.7) and (3.12) imply $y(i_0) \neq M$ and so (3.11) is true. Consequently, Theorem 1.2 guarantees that $(3.8)^m$ has a solution $y_m \in C(I_n)$ with $1/m \leq y_m(i) \leq M$ for $i \in I_n$ and y_m satisfies

$$\begin{aligned} (-1)^{n-p}\,\Delta^n y(k-p) &= f(k, y(k)), && k \in J_p, \\ y(0) &= \frac{1}{m}, \\ \Delta^i y(0) &= 0, && 1 \leq i \leq p-1, \\ y(T+n) &= \frac{1}{m}, \\ \Delta^i y(T+n-i) &= 0, && 1 \leq i \leq n-p-1, \end{aligned}$$

Next we obtain a sharper lower bound on y_m. Notice y_m satisfies

$$y_m(i) = \frac{1}{m} + \sum_{j=p}^{T+p} (-1)^{n-p} G(i,j)\, f\left(j, y_m(j)\right), \qquad \text{for } i \in I_n. \tag{3.13}$$

Also (3.3) guarantees the existence of a continuous function $\psi_M : J_p \to (0,\infty)$ with $f(i,u) \geq \psi_M(i)$ for $(i,u) \in J_p \times (0, M]$. This together with (3.13) yields

$$y_m(i) \geq \sum_{j=p}^{T+p} (-1)^{n-p} G(i,j)\, \psi_M(j) \equiv \Phi_M(i), \qquad \text{for } i \in J_p. \tag{3.14}$$

Clearly,

$$\{y_m\}_{m\in N_0} \text{ is a bounded family on } I_n. \tag{3.15}$$

The Arzela-Ascoli Theorem [2] guarantees the existence of a subsequence N of N_0 and a function $y \in C(I_n)$ with $y_n \to y$ in $C(I_n)$ as $n \to \infty$ through N. Also

$$y(0) = \cdots = y(p-1) = y(T+p+1) = \cdots = y(T+n) = 0.$$

Fix $i \in J_p$. Then y_m, $m \in N$, satisfies (3.13). Also

$$\Phi_M = \min_{i\in J_p} \Phi_M(i) \leq y_m(j) \leq M, \qquad \text{for } j \in J_p \text{ and } m \in N. \tag{3.16}$$

Let $m \to \infty$ through N in (3.13) to obtain

$$y(i) = \sum_{j=p}^{T+p} (-1)^{n-p} G(i,j)\, f(j, y(j)), \qquad \text{for } i \in J_p$$

and so $(-1)^{n-p}\, \Delta^n\, y(i-p) = f(i, y(i))$ for $i \in J_p$. Also notice that (3.16) implies $y(j) \geq \Phi_M > 0$ for $j \in J_p$. ∎

EXAMPLE 3.1. Consider the boundary value problem

$$\begin{aligned} (-1)^{n-p}\, \Delta^n\, y(k-p) &= \mu\left([y(k)]^{-\alpha} + A\,[y(k)]^{\beta} + B\right), && \text{for } k \in J_p, \\ \Delta^i\, y(0) &= 0, && 0 \leq i \leq p-1, \\ \Delta^i\, y(T+n-i) &= 0, && 0 \leq i \leq n-p-1, \end{aligned} \tag{3.17}$$

with $\alpha > 0$, $\beta \geq 0$, $A \geq 0$, $B \geq 0$, and $\mu > 0$. If

$$\mu < \frac{\theta^{\alpha}}{Q} \sup_{c\in(0,\infty)} \left(\frac{c^{\alpha+1}}{1 + A\, c^{\alpha+\beta} + B\, c^{\alpha}}\right) \tag{3.18}$$

(here θ is as in (1.10) and Q is as in the statement of Theorem 3.1) then (3.17) has a solution $y \in C(I_n)$ with $y(i) > 0$ for $i \in J_p$.

REMARK 3.2. If $\beta < 1$, then (3.18) is true for all $\mu > 0$.

The result follows immediately from Theorem 3.1 with $g(u) = \mu\, u^{-\alpha}$ and $h(u) = \mu\,[A\, u^{\beta} + B]$. Clearly, (3.1), (3.2), (3.3) (with $\psi_H = \mu\, H^{-\alpha}$), and (3.4) (with $K_\theta = \theta^{-\alpha}$) are satisfied. Also, (3.18) guarantees that (3.5) is true. Existence of a solution is now guaranteed from Theorem 3.1.

REFERENCES

1. R.P. Agarwal and P.J.Y. Wong, *Advanced Topics in Difference Equations*, Kluwer Academic Publishers, Dordrecht, (1997).
2. R.P. Agarwal, M. Bohner and P.J.Y. Wong, On eigenvalues and eigenfunctions of discrete conjugate boundary value problems, *Computers Math. Applic.* **38** (3/4), 159–183 (1999).
3. R.P. Agarwal, D. O'Regan and P.J.Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht (to appear).
4. D. O'Regan, *Existence Theory for Nonlinear Ordinary Differential Equations*, Kluwer Academic Publishers, Dordrecht, (1997).
5. P.J.Y. Wong and R.P. Agarwal, Extensions of continuous and discrete inequalities due to Eloe and Henderson, *Nonlinear Analysis* **34**, 479–487 (1998).
6. R.P. Agarwal and D. O'Regan, Singular discrete boundary value problems, *Appl. Math. Lett.* **12** (4), 127–131 (1999).
7. R.P. Agarwal and D. O'Regan, A fixed point approach for nonlinear discrete boundary value problems, *Advances in Difference Equations II,* Special Issue of *Computers Math. Applic.* (to appear).
8. R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, (1992).
9. P.W. Eloe, A generalizaion of concavity for finite differences, *Computers Math. Applic.* **36** (10–12), 109–113 (1998).
10. A. Lasota, A discrete boundary value problem, *Ann. Polon. Math.* **20**, 183–190 (1968).